

Approximation of conjugate functions by general linear operators of their Fourier series at the Lebesgue points

Włodzimierz Lenski and Bogdan Szal

University of Zielona Góra

Faculty of Mathematics, Computer Science and Econometrics

65-516 Zielona Góra, ul. Szafrana 4a, Poland

W.Lenski@wmie.uz.zgora.pl,

B.Szal @wmie.uz.zgora.pl

Abstract

The pointwise estimates of the deviations $\tilde{T}_{n,A,B}f(\cdot) - \tilde{f}(\cdot)$ and $\tilde{T}_{n,A,B}f(\cdot) - \tilde{f}(\cdot, \varepsilon)$ in terms of moduli of continuity $\tilde{w}.f$ and $\tilde{w}.f$ are proved. Analogical results on norm approximation with remarks and corollary are also given. These results generalized a theorem of Mittal [3, Theorem 1, p. 437]

Key words: Rate of approximation, summability of Fourier series,

2000 Mathematics Subject Classification: 42A24.

1 Introduction

Let L^p ($1 \leq p < \infty$) [$p = \infty$] be the class of all 2π -periodic real-valued functions integrable in the Lebesgue sense with p -th power [essentially bounded] over $Q = [-\pi, \pi]$ with the norm

$$\|f\| := \|f(\cdot)\|_{L^p} = \begin{cases} \left(\int_Q |f(t)|^p dt \right)^{1/p} & \text{when } 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{t \in Q} |f(t)| & \text{when } p = \infty \end{cases}$$

and consider the trigonometric Fourier series

$$Sf(x) := \frac{a_0(f)}{2} + \sum_{\nu=1}^{\infty} (a_{\nu}(f) \cos \nu x + b_{\nu}(f) \sin \nu x)$$

with the partial sums $S_k f$ and the conjugate one

$$\tilde{S}f(x) := \sum_{\nu=1}^{\infty} (a_{\nu}(f) \sin \nu x - b_{\nu}(f) \cos \nu x)$$

with the partial sums $\tilde{S}_k f$. We know that if $f \in L^1$ then

$$\tilde{f}(x) := -\frac{1}{\pi} \int_0^{\pi} \psi_x(t) \frac{1}{2} \cot \frac{t}{2} dt = \lim_{\epsilon \rightarrow 0^+} \tilde{f}(x, \epsilon),$$

where

$$\tilde{f}(x, \epsilon) := -\frac{1}{\pi} \int_{\epsilon}^{\pi} \psi_x(t) \frac{1}{2} \cot \frac{t}{2} dt$$

with

$$\psi_x(t) := f(x+t) - f(x-t),$$

exists for almost all x [5, Th.(3.1)IV].

Let $A := (a_{n,k})$ and $B := (b_{n,k})$ be infinite lower triangular matrices of real numbers such that

$$\begin{aligned} a_{n,k} &\geq 0 \text{ and } b_{n,k} \geq 0 \text{ when } k = 0, 1, 2, \dots, n, \\ a_{n,k} &= 0 \text{ and } b_{n,k} = 0 \text{ when } k > n, \end{aligned}$$

$$\sum_{k=0}^n a_{n,k} = 1 \text{ and } \sum_{k=0}^n b_{n,k} = 1, \text{ where } n = 0, 1, 2, \dots$$

Let define the general linear operators by the AB -transformation of $(S_k f)$ and $(\tilde{S}_k f)$ as follows

$$T_{n,A,B}f(x) := \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} S_k f(x) \quad (n = 0, 1, 2, \dots)$$

and

$$\tilde{T}_{n,A,B}f(x) := \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \tilde{S}_k f(x) \quad (n = 0, 1, 2, \dots). \quad (1)$$

As a measure of approximation of f, \tilde{f} and $\tilde{f}(\cdot, \epsilon)$ by the above quantities we use the pointwise moduli of continuity of f in the space L^1 defined by the formulas

$$w_x f(\delta) = \frac{1}{\delta} \int_0^{\delta} |\varphi_x(u)| du \text{ and } \tilde{w}_x f(\delta) = \frac{1}{\delta} \int_0^{\delta} |\psi_x(u)| du,$$

$$\bar{w}_x f(\delta) = \sup_{0 < t \leq \delta} \frac{1}{t} \int_0^t |\varphi_x(u)| du \text{ and } \widetilde{\bar{w}}_x f(\delta) = \sup_{0 < t \leq \delta} \frac{1}{t} \int_0^t |\psi_x(u)| du,$$

where

$$\varphi_x(t) := f(x+t) + f(x-t) - 2f(x).$$

It is clear that

$$\|w.f(\delta)\|_{L^p} \leq \omega f(\delta)_{L^p} \text{ and } \|\tilde{w}.f(\delta)\|_{L^p} \leq \tilde{\omega} f(\delta)_{L^p}, \quad (2)$$

where

$$\omega f(\delta)_{L^p} = \sup_{0 < t \leq \delta} \|\varphi.(t)\|_{L^p} \text{ and } \tilde{\omega} f(\delta)_{L^p} = \sup_{0 < t \leq \delta} \|\psi.(t)\|_{L^p}$$

are the classical moduli of continuity of f .

The deviation $T_{n,A,B}f - f$ with lower triangular infinite matrix A , defined by $a_{n,k} = \frac{1}{n+1}$ when $k = 0, 1, 2, \dots, n$ and $a_{n,k} = 0$ when $k > n$, was estimated by M. L. Mittal as follows:

Theorem A. [3, Theorem 1, p. 437] *If entries of matrix A satisfy the conditions*

$$a_{n,n-k} - a_{n+1,n+1-k} \geq 0 \text{ for } 0 \leq k \leq n,$$

$$\sum_{k=0}^l (k+1) |a_{r,r-k} - a_{r,r-k-1}| = O\left(\sum_{k=r-l}^r a_{r,k}\right) \text{ for } 0 \leq l \leq r \leq n,$$

$$\sum_{k=0}^r (k+1) |(a_{r,r-k} - a_{r+1,r+1-k}) - (a_{r,r-k-1} - a_{r+1,r-k})|$$

$$= O\left(\frac{1}{r+1}\right) \text{ for } 0 \leq r \leq n$$

and

$$a_{n,0} - a_{n+1,1} = O\left((n+1)^{-2}\right)$$

then, for x such that

$$\frac{1}{t} \int_0^t |\varphi_x(u)| du = o_x(1) \text{ as } t \rightarrow 0+,$$

we have the relation

$$\frac{1}{n+1} \sum_{r=0}^n \sum_{k=0}^r a_{r,k} S_k f(x) - f(x) = o_x(1) \text{ as } n \rightarrow \infty.$$

The deviation $T_{n,A,B}f - f$ was also estimated in our earlier paper as follows:

Theorem B. [2] *Let $f \in L^1$. If entries of our matrices satisfy the conditions*

$$a_{n,n} \ll \frac{1}{n+1},$$

$$\frac{1}{s+1} \sum_{r=0}^s a_{n,r} \ll a_{n,s} \text{ for } 0 \leq s \leq n$$

and

$$|a_{n,r}b_{r,r-l} - a_{n,r+1}b_{r+1,r+1-l}| \ll \frac{a_{n,r}}{(r+1)^2} \text{ for } 0 \leq l \leq r \leq n-1,$$

then

$$|T_{n,A,B}f(x) - f(x)| \ll \sum_{r=0}^n a_{n,r} \left[\frac{1}{r+1} \sum_{k=0}^r \bar{w}_x f\left(\frac{\pi}{k+1}\right) \right],$$

$$\|T_{n,A,B}f(\cdot) - f(\cdot)\|_{L^p} \ll \sum_{r=0}^n a_{n,r} \left[\frac{1}{r+1} \sum_{k=0}^r \omega f\left(\frac{\pi}{k+1}\right)_{L^p} \right],$$

for every natural n and all real x .

In the case of conjugate functions the deviation $\sum_{k=0}^n a_{n,k} \tilde{S}_k f(x) - \tilde{f}(x)$ was considered by M. L. Mittal, B. E. Rhoades and V. N. Mishra in [4] in the following way

Theorem C. [4, Theorem 3.1] *Let $f \in L^p$ ($p \geq 1$) such that*

$$\int_0^{2\pi} |[f(x+t) - f(x)] \sin^\beta x|^p dx = O(\xi(t)) \quad (\beta \geq 0)$$

provided that $\xi(t)$ is positive, increasing function of t satisfying the conditions

$$\left\{ \int_0^{\pi/n} \left(\frac{t |\psi_x(t)|}{\xi(t)} \right)^p \sin^{\beta p} t dt \right\}^{1/p} = O_x \left(\frac{1}{n} \right)$$

$$\left\{ \int_{\pi/n}^\pi \left(\frac{t^{-\delta} |\psi_x(t)|}{\xi(t)} \right)^p dt \right\}^{1/p} = O_x(n^\delta)$$

uniformly in x , where $\delta < 1/p$, but the function $\frac{\xi(t)}{t}$ decreasing in t . If the entries of matrix A satisfy the condition

$$\sum_{k=0}^l (k+1) |a_{r,r-k} - a_{r,r-k-1}| = O \left(\sum_{k=r-l}^r a_{r,k} \right) \quad \text{for } 0 \leq l \leq r \leq n,$$

then

$$\left\| \sum_{k=0}^n a_{n,k} \tilde{S}_k f(\cdot) - \tilde{f}(\cdot) \right\|_{L^p} = O \left(n^{\beta + \frac{1}{p}} \xi \left(\frac{1}{n} \right) \right).$$

In our theorems we will consider the deviations $\tilde{T}_{n,A,B}f(\cdot) - \tilde{f}(\cdot)$ and $\tilde{T}_{n,A,B}f(\cdot) - \tilde{f}(\cdot, \varepsilon)$ with the mean (1) introduced at the begin and we will present the estimates of the above type. Consequently, we also give some results on norm approximation and some remarks. Finally, we will derive a corollary.

We shall write $I_1 \ll I_2$ if there exists a positive constant K , sometimes depend on some parameters, such that $I_1 \leq KI_2$.

2 Statement of the results

We start with our main results on the degrees of pointwise summability.

Theorem 1 *Let $f \in L^1$. If entries of our matrices satisfy the conditions*

$$a_{n,n} \ll \frac{1}{n+1}, \quad (3)$$

$$\frac{1}{s+1} \sum_{r=0}^s a_{n,r} \ll a_{n,s} \text{ for } 0 \leq s \leq n \quad (4)$$

and

$$|a_{n,r} b_{r,r-l} - a_{n,r+1} b_{r+1,r+1-l}| \ll \frac{a_{n,r}}{(r+1)^2} \text{ for } 0 \leq l \leq r \leq n-1, \quad (5)$$

then

$$\left| \tilde{T}_{n,A,B} f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \right| \ll \sum_{r=0}^n a_{n,r} \left[\frac{1}{r+1} \sum_{k=0}^r \tilde{w}_x f\left(\frac{\pi}{k+1}\right) \right] \quad (6)$$

and under the additional condition

$$\frac{1}{\pi} \int_0^{\frac{\pi}{n+1}} \frac{|\psi_x(t)|}{t} dt \ll \tilde{w}_x f\left(\frac{\pi}{n+1}\right), \quad (7)$$

$$\left| \tilde{T}_{n,A,B} f(x) - \tilde{f}(x) \right| \ll \sum_{r=0}^n a_{n,r} \left[\frac{1}{r+1} \sum_{k=0}^r \tilde{w}_x f\left(\frac{\pi}{k+1}\right) \right] \quad (8)$$

for every natural n and all considered real x .

Remark 1 *We can observe that the proof of Theorem 1 yields the following more precise estimate*

$$\left. \begin{aligned} & \left| \tilde{T}_{n,A,B} f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \right| \\ & \left| \tilde{T}_{n,A,B} f(x) - \tilde{f}(x) \right| \end{aligned} \right\} \ll \sum_{r=0}^n \left(a_{n,r} + \sum_{k=1}^r \frac{a_{n,k}}{r+1} \right) \left[\frac{1}{r+1} \sum_{k=0}^r \tilde{w}_x f\left(\frac{\pi}{k+1}\right) \right] \\ + \left[\frac{1}{n+1} \sum_{k=0}^n \tilde{w}_x f\left(\frac{\pi}{k+1}\right) \right], \quad (9)$$

without assumption (4). In this case we can obtain the relation

$$\left. \begin{aligned} & \left| \tilde{T}_{n,A,B} f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \right| \\ & \left| \tilde{T}_{n,A,B} f(x) - \tilde{f}(x) \right| \end{aligned} \right\} = o_x(1)$$

under weaker assumption $\sum_{r=0}^n \sum_{k=0}^r \frac{a_{n,k}}{r+1} = O(1)$ instead of (4) with (3), (5) and $\tilde{w}_x f(\delta) = o_x(1)$.

If we suppose at the beginning that the matrix A is such that $a_{n,k} = \frac{1}{n+1}$ when $k = 0, 1, 2, \dots, n$ and $a_{n,k} = 0$ when $k > n$, then we can yet reduce our assumptions.

Theorem 2 *Let $f \in L^1$ and let entries of the matrix B satisfy the condition*

$$|b_{r,r-l} - b_{r+1,r+1-l}| \ll \frac{1}{(r+1)^2} \text{ for } 0 \leq l \leq r. \quad (10)$$

Then

$$\left| \tilde{T}_{n,(\frac{1}{n+1}),B} f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \right| \ll \frac{1}{n+1} \sum_{r=0}^n \left[\frac{1}{r+1} \sum_{k=0}^r \tilde{w}_x f\left(\frac{\pi}{k+1}\right) \right]$$

and under (7)

$$\left| \tilde{T}_{n,(\frac{1}{n+1}),B} f(x) - \tilde{f}(x) \right| \ll \frac{1}{n+1} \sum_{r=0}^n \left[\frac{1}{r+1} \sum_{k=0}^r \tilde{w}_x f\left(\frac{\pi}{k+1}\right) \right], \quad (11)$$

for every natural n and all considered real x .

Remark 2 *Analyzing the proof of Theorem 1 with $A = \left(\frac{1}{n+1}\right)$ we can see that under the following additional assumption on the entries of the matrix B*

$$\sum_{r=s}^{n-1} \sum_{k=s}^r |b_{r,r-k} - b_{r+1,r+1-k}| \ll 1 \text{ for } 0 < s \leq r \leq n-1,$$

we obtain the more precise estimate

$$\left| \tilde{T}_{n,(\frac{1}{n+1}),B} f(x) - \tilde{f}(x) \right| \ll \frac{1}{n+1} \sum_{r=0}^n \tilde{w}_x f\left(\frac{\pi}{r+1}\right)$$

than (11). This additional assumption as well (10) are fulfilled if $b_{n,k} = 0$ for $k = 0, 1, 2, \dots, n-1, n+1, \dots$ and $b_{n,n} = 1$. Thus we have analogue of the well known classical result of S. Aljančič, R. Bojanic and M. Tomić [1]. The analogical remark we can prepare with respect to Theorem 1.

Finally, we formulate the results on the estimates of L^p norm of the deviation considered above.

Theorem 3 *Let $f \in L^p$. Under the assumptions of Theorem 1 on the entries of matrices A and B and $\tilde{\omega}f$, we have*

$$\left\{ \begin{array}{l} \left\| \tilde{T}_{n,A,B} f(\cdot) - \tilde{f}\left(\cdot, \frac{\pi}{n+1}\right) \right\|_{L^p} \\ \left\| \tilde{T}_{n,A,B} f(\cdot) - \tilde{f}(\cdot) \right\|_{L^p} \end{array} \right\} \ll \sum_{r=0}^n a_{n,r} \left[\frac{1}{r+1} \sum_{k=0}^r \tilde{\omega}f\left(\frac{\pi}{k+1}\right)_{L^p} \right],$$

for every natural n .

Theorem 4 Let $f \in L^p$, then, under the assumptions of Theorem 2 on the entries of matrix B and $\tilde{\omega}f$, we have

$$\left\{ \begin{array}{l} \left\| \tilde{T}_{n,(\frac{1}{n+1}),B} f(\cdot) - \tilde{f}\left(\cdot, \frac{\pi}{n+1}\right) \right\|_{L^p} \\ \left\| \tilde{T}_{n,(\frac{1}{n+1}),B} f(\cdot) - \tilde{f}(\cdot) \right\|_{L^p} \end{array} \right\} \ll \frac{1}{n+1} \sum_{r=0}^n \left[\frac{1}{r+1} \sum_{k=0}^r \tilde{\omega}f\left(\frac{\pi}{k+1}\right)_{L^p} \right],$$

for every natural n .

Corollary 1 Under the assumptions of our Theorems we have the relations

$$\left\{ \begin{array}{l} \left| \tilde{T}_{n,A,B} f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \right| \\ \left| \tilde{T}_{n,A,B} f(x) - \tilde{f}(x) \right| \end{array} \right\} = o_x(1), \quad \text{a.e. in } x,$$

$$\left\{ \begin{array}{l} \left| \tilde{T}_{n,(\frac{1}{n+1}),B} f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \right| \\ \left| \tilde{T}_{n,(\frac{1}{n+1}),B} f(x) - \tilde{f}(x) \right| \end{array} \right\} = o_x(1), \quad \text{a.e. in } x,$$

such that $\tilde{w}_x f(t) = o_x(1)$ as $t \rightarrow 0+$, and

$$\left\{ \begin{array}{l} \left\| \tilde{T}_{n,A,B} f(\cdot) - \tilde{f}\left(\cdot, \frac{\pi}{n+1}\right) \right\|_{L^p} \\ \left\| \tilde{T}_{n,A,B} f(\cdot) - \tilde{f}(\cdot) \right\|_{L^p} \end{array} \right\} = o(1),$$

$$\left\{ \begin{array}{l} \left\| \tilde{T}_{n,(\frac{1}{n+1}),B} f(\cdot) - \tilde{f}\left(\cdot, \frac{\pi}{n+1}\right) \right\|_{L^p} \\ \left\| \tilde{T}_{n,(\frac{1}{n+1}),B} f(\cdot) - \tilde{f}(\cdot) \right\|_{L^p} \end{array} \right\} = o(1),$$

when $n \rightarrow \infty$.

3 Auxiliary results

We begin this section by some notations following A. Zygmund [5, Section 5 of Chapter II].

It is clear that

$$\tilde{S}_k f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \tilde{D}_k(t) dt,$$

and

$$\tilde{T}_{n,A,B} f(x) = -\frac{1}{\pi} \int_{-\pi}^{\pi} f(x+t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \tilde{D}_k(t) dt,$$

where

$$\tilde{D}_k(t) = \sum_{\nu=0}^k \sin \nu t = \frac{\cos \frac{t}{2} - \cos \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} = \frac{2 \sin \frac{kt}{2} \sin \frac{(k+1)t}{2}}{2 \sin \frac{t}{2}}.$$

Hence

$$\begin{aligned}\widetilde{T}_{n,A,B}f(x) - \widetilde{f}\left(x, \frac{\pi}{n+1}\right) &= -\frac{1}{\pi} \int_0^{\frac{\pi}{n+1}} \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \widetilde{D}_k(t) dt \\ &\quad + \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \widetilde{D}_k^\circ(t) dt\end{aligned}$$

and

$$\widetilde{T}_{n,A,B}f(x) - \widetilde{f}(x) = \frac{1}{\pi} \int_0^{\pi} \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \widetilde{D}_k^\circ(t) dt,$$

where

$$\widetilde{D}_k^\circ(t) = \frac{1}{2} \cot \frac{t}{2} - \sum_{\nu=0}^k \sin \nu t = \frac{\cos \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}}.$$

Now, we formulate some estimates for the conjugate Dirichlet kernels.

Lemma 1 (see [5, Section 5 of Chapter II, p. 51]) *If $0 < |t| \leq \pi/2$, then*

$$\left| \widetilde{D}_k^\circ(t) \right| \leq \frac{\pi}{2|t|} \quad \text{and} \quad \left| \widetilde{D}_k(t) \right| \leq \frac{\pi}{|t|}$$

but for any real t we have

$$\left| \widetilde{D}_k(t) \right| \leq \frac{1}{2} k(k+1) |t| \quad \text{and} \quad \left| \widetilde{D}_k(t) \right| \leq k+1.$$

More complicated estimates we give with proofs.

Lemma 2 *Let $f \in L^1$. The following inequalities*

$$\begin{aligned}\widetilde{w}_x f\left(\frac{\pi}{n+1}\right) &\leq 2 \frac{1}{n+1} \sum_{r=0}^n \widetilde{w}_x f\left(\frac{\pi}{r+1}\right) \\ \widetilde{\widetilde{w}}_x f\left(\frac{\pi}{n+1}\right) &\leq \frac{1}{n+1} \sum_{r=0}^n \widetilde{\widetilde{w}}_x f\left(\frac{\pi}{r+1}\right)\end{aligned}$$

hold for every naturals n and all real x .

Proof. The proof of the first inequality follows by the easy account

$$\begin{aligned}\widetilde{w}_x f\left(\frac{\pi}{n+1}\right) &= \frac{n+1}{\pi} \int_0^{\frac{\pi}{n+1}} |\psi_x(u)| du \sum_{r=0}^n \frac{2(r+1)}{(n+1)(n+2)} \\ &= \frac{2}{n+1} \sum_{r=0}^n \frac{(r+1)(n+1)}{\pi(n+2)} \int_0^{\frac{\pi}{n+1}} |\psi_x(u)| du \\ &\leq \frac{2}{n+1} \sum_{r=0}^n \frac{r+1}{\pi} \int_0^{\frac{\pi}{r+1}} |\psi_x(u)| du \\ &= 2 \frac{1}{n+1} \sum_{r=0}^n \widetilde{w}_x f\left(\frac{\pi}{r+1}\right).\end{aligned}$$

The second inequality is evident, therefore our proof is completed. ■

4 Proofs of the results

4.1 Proof of Theorem 1

First, we prove the relation (6). The main idea of the proof based on the method used in [2, Proof of Theorem 1]. Let

$$\begin{aligned}
\tilde{T}_{n,A,B}f(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) &= \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \tilde{S}_k(x) - \tilde{f}\left(x, \frac{\pi}{n+1}\right) \\
&= -\frac{1}{\pi} \int_0^{\frac{\pi}{n+1}} \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \tilde{D}_k(t) dt + \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \tilde{D}_k^\circ(t) dt \\
&= I_1 + I_2.
\end{aligned}$$

Further, by Lemma 1,

$$\begin{aligned}
|I_1| &\leq \frac{1}{\pi} \int_0^{\frac{\pi}{n+1}} |\psi_x(t)| \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \left| \frac{2 \sin \frac{kt}{2} \sin \frac{(k+1)t}{2}}{2 \sin \frac{t}{2}} \right| dt \\
&\leq \frac{1}{n+1} \tilde{w}_x f\left(\frac{\pi}{n+1}\right) \sum_{r=0}^n \sum_{k=0}^r (k+1) a_{n,r} b_{r,k} \\
&\leq \tilde{w}_x f\left(\frac{\pi}{n+1}\right) \sum_{r=0}^n a_{n,r} \leq \sum_{r=0}^n a_{n,r} \tilde{w}_x f\left(\frac{\pi}{r+1}\right) \\
&\ll \sum_{r=0}^n a_{n,r} \left[\frac{1}{r+1} \sum_{k=0}^r \tilde{w}_x f\left(\frac{\pi}{k+1}\right) \right]
\end{aligned}$$

and using the inequality: $\sin t \leq \frac{2t}{\pi}$ for $0 < t \leq \frac{\pi}{2}$

$$\begin{aligned}
|I_2| &\leq \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \left| \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \frac{\cos \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} \right| dt \\
&\ll \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \left| \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \cos \frac{(2k+1)t}{2} \right| dt \\
&\leq \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \left| \sum_{r=0}^{\tau-1} \sum_{k=0}^r a_{n,r} b_{r,k} \cos \frac{(2k+1)t}{2} \right| dt \\
&\quad + \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \left| \sum_{r=\tau}^n \sum_{k=0}^{\tau-1} a_{n,r} b_{r,r-k} \cos \frac{(2r-2k+1)t}{2} \right| dt \\
&\quad + \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \left| \sum_{r=\tau}^n \sum_{k=\tau}^r a_{n,r} b_{r,r-k} \cos \frac{(2r-2k+1)t}{2} \right| dt
\end{aligned}$$

$$= I_{21} + I_{22} + I_{23},$$

where $\tau = \left\lceil \frac{\pi}{t} \right\rceil$ for $t \in (0, \pi]$. Now, we shall estimate the integrals I_{21} , I_{22} and I_{23} . So,

$$\begin{aligned} I_{21} &\leq \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \sum_{r=0}^{\tau-1} \sum_{k=0}^r a_{n,r} b_{r,k} dt \\ &= \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \sum_{r=0}^{\tau-1} a_{n,r} dt \leq 2\pi \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t^2} \sum_{r=0}^{\tau-1} \frac{a_{n,r}}{\tau+1} dt \\ &= 2\pi \sum_{s=1}^n \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \frac{|\psi_x(t)|}{t^2} \sum_{r=0}^{\tau-1} \frac{a_{n,r}}{\tau+1} dt \leq 2\pi \sum_{s=1}^n \sum_{r=0}^s \frac{a_{n,r}}{s+1} \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \frac{|\psi_x(t)|}{t^2} dt \end{aligned}$$

and integrating by parts

$$\begin{aligned} I_{21} &\leq 2\pi \sum_{s=1}^n \sum_{r=0}^s \frac{a_{n,r}}{s+1} \left\{ \left[\frac{1}{t^2} \int_0^t |\psi_x(u)| du \right]_{t=\frac{\pi}{s+1}}^{\frac{\pi}{s}} + 2 \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \left[\frac{1}{t^3} \int_0^t |\psi_x(u)| du \right] dt \right\} \\ &= 2\pi \sum_{s=0}^{n-1} \sum_{r=0}^s \frac{a_{n,r}}{s+2} \left[\frac{1}{t^2} \int_0^t |\psi_x(u)| du \right]_{t=\frac{\pi}{s+2}}^{\frac{\pi}{s+1}} + 2\pi \sum_{s=1}^n \frac{a_{n,s}}{s+1} \left[\frac{1}{t^2} \int_0^t |\psi_x(u)| du \right]_{t=\frac{\pi}{s+1}}^{\frac{\pi}{s}} \\ &\quad + 4\pi \sum_{s=1}^n \sum_{r=0}^s \frac{a_{n,r}}{s+1} \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \left[\frac{1}{t^3} \int_0^t |\psi_x(u)| du \right] dt \\ &= 2\pi \sum_{s=0}^{n-1} \sum_{r=0}^s \frac{a_{n,r}}{s+2} \left[\frac{\tilde{w}_x f\left(\frac{\pi}{s+1}\right)}{\frac{\pi}{s+1}} - \frac{\tilde{w}_x f\left(\frac{\pi}{s+2}\right)}{\frac{\pi}{s+2}} \right] \\ &\quad + 2\pi \sum_{s=1}^n \frac{a_{n,s}}{s+1} \left[\frac{\tilde{w}_x f\left(\frac{\pi}{s}\right)}{\frac{\pi}{s}} - \frac{\tilde{w}_x f\left(\frac{\pi}{s+1}\right)}{\frac{\pi}{s+1}} \right] \\ &\quad + 4\pi \sum_{s=1}^n \sum_{r=0}^s \frac{a_{n,r}}{s+1} \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \left[\frac{1}{t^3} \int_0^t |\psi_x(u)| du \right] dt \\ &\leq 2\pi \sum_{s=0}^{n-1} \sum_{r=0}^s \frac{a_{n,r}}{r+1} \left[\frac{\tilde{w}_x f\left(\frac{\pi}{s+1}\right)}{\frac{\pi}{s+1}} - \frac{\tilde{w}_x f\left(\frac{\pi}{s+2}\right)}{\frac{\pi}{s+2}} \right] + 2\pi \sum_{s=1}^n \frac{a_{n,s}}{s+1} \frac{\tilde{w}_x f\left(\frac{\pi}{s}\right)}{\frac{\pi}{s}} \\ &\quad + 4\pi \sum_{s=1}^n \sum_{r=0}^s \frac{a_{n,r}}{s+1} \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \left[\frac{1}{t^3} \int_0^t |\psi_x(u)| du \right] dt. \end{aligned}$$

Changing the order of summation we get

$$\begin{aligned}
I_{21} &\ll \sum_{r=0}^{n-1} \frac{a_{n,r}}{r+1} \sum_{s=r}^{n-1} \left[\frac{\tilde{w}_x f\left(\frac{\pi}{s+1}\right)}{\frac{\pi}{s+1}} - \frac{\tilde{w}_x f\left(\frac{\pi}{s+2}\right)}{\frac{\pi}{s+2}} \right] + \sum_{s=1}^n a_{n,s} \tilde{w}_x f\left(\frac{\pi}{s}\right) \\
&\quad + \sum_{s=1}^n \sum_{r=0}^s \frac{a_{n,r}}{s+1} \tilde{w}_x f\left(\frac{\pi}{s}\right) \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \frac{1}{t^2} dt \\
&= \sum_{r=0}^{n-1} \frac{a_{n,r}}{r+1} \left[\frac{\tilde{w}_x f\left(\frac{\pi}{r+1}\right)}{\frac{\pi}{r+1}} - \frac{\tilde{w}_x f\left(\frac{\pi}{n+1}\right)}{\frac{\pi}{n+1}} \right] + \sum_{s=1}^n a_{n,s} \tilde{w}_x f\left(\frac{\pi}{s}\right) \\
&\quad + \sum_{s=1}^n \sum_{r=0}^s \frac{a_{n,r}}{s+1} \tilde{w}_x f\left(\frac{\pi}{s}\right) \left(\frac{1}{\frac{\pi}{s+1}} - \frac{1}{\frac{\pi}{s}} \right) \\
&\ll \sum_{r=0}^{n-1} \frac{a_{n,r}}{r+1} \frac{\tilde{w}_x f\left(\frac{\pi}{r+1}\right)}{\frac{\pi}{r+1}} + \sum_{s=1}^n a_{n,s} \tilde{w}_x f\left(\frac{\pi}{s}\right) + \sum_{s=1}^n \sum_{r=0}^s \frac{a_{n,r}}{s+1} \tilde{w}_x f\left(\frac{\pi}{s}\right) \\
&\leq \frac{1}{\pi} \sum_{r=0}^n a_{n,r} \tilde{w}_x f\left(\frac{\pi}{r+1}\right) + \sum_{s=1}^n \left(a_{n,s} + \sum_{r=0}^s \frac{a_{n,r}}{s+1} \right) \tilde{w}_x f\left(\frac{\pi}{s}\right) \\
&\ll \sum_{r=0}^n a_{n,r} \left[\frac{1}{r+1} \sum_{s=0}^r \tilde{w}_x f\left(\frac{\pi}{s+1}\right) \right] \\
&\quad + \sum_{s=1}^n \left(a_{n,s} + \sum_{r=0}^s \frac{a_{n,r}}{s+1} \right) \left[\frac{1}{s} \sum_{r=1}^s \tilde{w}_x f\left(\frac{\pi}{r}\right) \right] \\
&= \sum_{r=0}^n a_{n,r} \left[\frac{1}{r+1} \sum_{s=0}^r \tilde{w}_x f\left(\frac{\pi}{s+1}\right) \right] \\
&\quad + \sum_{s=1}^n \left(a_{n,s} + \sum_{r=0}^s \frac{a_{n,r}}{s+1} \right) \left[\frac{1}{s} \sum_{r=0}^{s-1} \tilde{w}_x f\left(\frac{\pi}{r+1}\right) \right] \\
&\ll \sum_{r=0}^n \left(a_{n,r} + \sum_{s=0}^r \frac{a_{n,s}}{r+1} \right) \left[\frac{1}{r+1} \sum_{s=0}^r \tilde{w}_x f\left(\frac{\pi}{s+1}\right) \right].
\end{aligned}$$

By the assumption (4)

$$I_{21} \ll \sum_{r=0}^n a_{n,r} \left[\frac{1}{r+1} \sum_{s=0}^r \tilde{w}_x f\left(\frac{\pi}{s+1}\right) \right] \leq \sum_{r=0}^n a_{n,r} \left[\frac{1}{r+1} \sum_{k=0}^r \tilde{w}_x f\left(\frac{\pi}{k+1}\right) \right].$$

Using Abel's transformation in I_{22} and conditions (3), and (5), we obtain

$$I_{22} = \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \left| \sum_{k=0}^{\tau-1} \left(\sum_{r=\tau}^{n-1} [a_{n,r} b_{r,r-k} - a_{n,r+1} b_{r+1,r+1-k}] \sum_{l=\tau}^r \cos \frac{(2l-2k+1)t}{2} \right) \right|$$

$$\begin{aligned}
& + a_{n,n} b_{n,n-k} \sum_{l=\tau}^n \cos \frac{(2l-2k+1)t}{2} \Bigg| dt \\
& \leq \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \sum_{k=0}^{\tau-1} \left[\sum_{r=\tau}^{n-1} |a_{n,r} b_{r,r-k} - a_{n,r+1} b_{r+1,r+1-k}| \tau + a_{n,n} b_{n,n-k} \tau \right] dt \\
& \leq \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \sum_{k=0}^{\tau} \tau \left[\sum_{r=\tau}^{n-1} \frac{a_{n,r}}{(r+1)^2} + a_{n,n} b_{n,n-k} \right] dt \\
& \leq \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \tau \left[(\tau+1) \sum_{r=\tau}^{n-1} \frac{a_{n,r}}{(r+1)^2} + a_{n,n} \sum_{k=0}^{\tau} b_{n,n-k} \right] dt \\
& \leq \pi \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t^2} \left[\sum_{r=\tau}^{n-1} \frac{a_{n,r}}{r+1} + a_{n,n} \right] dt \\
& = \pi \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t^2} \sum_{r=\tau}^{n-1} \frac{a_{n,r}}{r+1} dt + \pi a_{n,n} \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t^2} dt \\
& \leq \pi \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t^2} \sum_{r=\tau}^n \frac{a_{n,r}}{r+1} dt + \pi a_{n,n} \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t^2} dt \\
& \leq \pi \sum_{s=1}^n \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \frac{|\psi_x(t)|}{t^2} \sum_{r=\tau}^n \frac{a_{n,r}}{r+1} dt + \frac{\pi}{n+1} \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t^2} dt.
\end{aligned}$$

Since the sequence $\left(\sum_{r=k}^n \frac{a_{n,r}}{r+1} \right)$ is nonincreasing in k , after changing the order of summation, we have

$$\begin{aligned}
I_{22} & \leq \pi \sum_{s=1}^n \sum_{r=s}^n \frac{a_{n,r}}{r+1} \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \frac{|\psi_x(t)|}{t^2} dt + \frac{\pi}{n+1} \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t^2} dt \\
& \leq \pi \sum_{s=1}^n \sum_{r=s}^n \frac{a_{n,r}}{r+1} \left\{ \left[\frac{1}{t^2} \int_0^t |\psi_x(u)| du \right]_{t=\frac{\pi}{s+1}}^{\frac{\pi}{s}} + 2 \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \left[\frac{1}{t^3} \int_0^t |\psi_x(u)| du \right] dt \right\} \\
& \quad + \frac{\pi}{n+1} \left[\frac{\tilde{w}_x f(\pi)}{\pi} + 2 \int_{\frac{\pi}{n+1}}^{\pi} \left[\frac{1}{t^3} \int_0^t |\psi_x(u)| du \right] dt \right] \\
& \ll \sum_{s=1}^n \sum_{r=s}^n \frac{a_{n,r}}{r+1} \left\{ \left[\frac{\tilde{w}_x f\left(\frac{\pi}{s}\right)}{\frac{\pi}{s}} - \frac{\tilde{w}_x f\left(\frac{\pi}{s+1}\right)}{\frac{\pi}{s+1}} \right] + \tilde{w}_x f\left(\frac{\pi}{s}\right) \int_{\frac{\pi}{s+1}}^{\frac{\pi}{s}} \frac{1}{t^2} dt \right\} \\
& \quad + \frac{1}{n+1} \sum_{s=1}^n \tilde{w}_x f\left(\frac{\pi}{s+1}\right)
\end{aligned}$$

$$\begin{aligned}
& \ll \sum_{r=1}^n \frac{a_{n,r}}{r+1} \sum_{s=1}^r \left\{ \left[\frac{\tilde{w}_x f\left(\frac{\pi}{s}\right)}{\frac{\pi}{s}} - \frac{\tilde{w}_x f\left(\frac{\pi}{s+1}\right)}{\frac{\pi}{s+1}} \right] + \tilde{w}_x f\left(\frac{\pi}{s}\right) \right\} \\
& \quad + \frac{1}{n+1} \sum_{s=1}^n \tilde{w}_x f\left(\frac{\pi}{s+1}\right) \\
& = \sum_{r=1}^n \frac{a_{n,r}}{r+1} \left\{ \left[\frac{\tilde{w}_x f(\pi)}{\pi} - \frac{\tilde{w}_x f\left(\frac{\pi}{r+1}\right)}{\frac{\pi}{r+1}} \right] + \sum_{s=1}^r \tilde{w}_x f\left(\frac{\pi}{s}\right) \right\} \\
& \quad + \frac{1}{n+1} \sum_{s=1}^n \tilde{w}_x f\left(\frac{\pi}{s+1}\right) \\
& \leq \sum_{r=1}^n \frac{a_{n,r}}{r+1} \left\{ \frac{\tilde{w}_x f(\pi)}{\pi} + \sum_{s=1}^r \tilde{w}_x f\left(\frac{\pi}{s}\right) \right\} + \frac{1}{n+1} \sum_{s=1}^n \tilde{w}_x f\left(\frac{\pi}{s+1}\right) \\
& \ll \sum_{r=1}^n \frac{a_{n,r}}{r+1} \sum_{s=0}^r \tilde{w}_x f\left(\frac{\pi}{s+1}\right) + \frac{1}{n+1} \sum_{s=0}^n \tilde{w}_x f\left(\frac{\pi}{s+1}\right) \\
& \ll \sum_{r=0}^n a_{n,r} \frac{1}{r+1} \sum_{s=0}^r \tilde{w}_x f\left(\frac{\pi}{s+1}\right)
\end{aligned}$$

and

$$\begin{aligned}
I_{23} & \leq \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \left| \sum_{r=\tau}^n \sum_{k=\tau}^r a_{n,r} b_{r,r-k} \cos \frac{(2r-2k+1)t}{2} \right| dt \\
& = \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \left| \sum_{k=\tau}^n \sum_{r=k}^n a_{n,r} b_{r,r-k} \cos \frac{(2r-2k+1)t}{2} \right| dt \\
& = \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \left| \sum_{k=\tau}^n \left[\sum_{r=k}^{n-1} (a_{n,r} b_{r,r-k} - a_{n,r+1} b_{r+1,r+1-k}) \sum_{l=k}^r \cos \frac{(2l-2k+1)t}{2} \right. \right. \\
& \quad \left. \left. + a_{n,n} b_{n,n-k} \sum_{l=k}^n \cos \frac{(2l-2k+1)t}{2} \right] \right| dt \\
& \leq \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \sum_{k=\tau}^n \left[\sum_{r=k}^{n-1} |a_{n,r} b_{r,r-k} - a_{n,r+1} b_{r+1,r+1-k}| \tau + a_{n,n} b_{n,n-k} \tau \right] dt \\
& \leq \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t} \tau \left[\sum_{r=\tau}^n \sum_{k=\tau}^r |a_{n,r} b_{r,r-k} - a_{n,r+1} b_{r+1,r+1-k}| + a_{n,n} \sum_{k=\tau}^n b_{n,n-k} \right] dt \\
& \ll \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t^2} \left[\sum_{r=\tau}^n \sum_{k=\tau}^r \frac{a_{n,r}}{(r+1)^2} + a_{n,n} \sum_{k=0}^{n-\tau} b_{n,k} \right] dt
\end{aligned}$$

$$\begin{aligned}
&\leq \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t^2} \left[\sum_{r=\tau}^n \frac{a_{n,r}}{(r+1)^2} (r-\tau+1) + a_{n,n} \sum_{k=0}^n b_{n,k} \right] dt \\
&\leq \int_{\frac{\pi}{n+1}}^{\pi} \frac{|\psi_x(t)|}{t^2} \left[\sum_{r=\tau}^n \frac{a_{n,r}}{r+1} + a_{n,n} \right] dt.
\end{aligned}$$

Further, the same calculation, as that in the estimate of I_{22} , gives the inequality

$$I_{23} \ll \sum_{r=0}^n a_{n,r} \frac{1}{r+1} \sum_{s=0}^r \tilde{w}_x f\left(\frac{\pi}{s+1}\right).$$

Collecting these estimates we obtain the desired estimate.

Now, we prove the relation (8). Let

$$\begin{aligned}
\tilde{T}_{n,A,B}(x) - \tilde{f}(x) &= \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \tilde{S}_k f(x) - \tilde{f}(x) \\
&= \frac{1}{\pi} \int_0^{\pi} \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \tilde{D}_k^{\circ}(t) dt \\
&= \left(\frac{1}{\pi} \int_0^{\frac{\pi}{n+1}} + \frac{1}{\pi} \int_{\frac{\pi}{n+1}}^{\pi} \right) \psi_x(t) \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \tilde{D}_k^{\circ}(t) dt \\
&= J_1 + J_2.
\end{aligned}$$

Further

$$\begin{aligned}
|J_1| &\leq \frac{1}{\pi} \int_0^{\frac{\pi}{n+1}} |\psi_x(t)| \left| \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \frac{\cos \frac{(2k+1)t}{2}}{2 \sin \frac{t}{2}} \right| dt \\
&\leq \frac{1}{2} \int_0^{\frac{\pi}{n+1}} \frac{|\psi_x(t)|}{t} \left| \sum_{r=0}^n \sum_{k=0}^r a_{n,r} b_{r,k} \cos \frac{(2k+1)t}{2} \right| dt \\
&\leq \frac{1}{2} \int_0^{\frac{\pi}{n+1}} \frac{|\psi_x(t)|}{t} dt \ll \tilde{w}_x f\left(\frac{\pi}{n+1}\right) \\
&= \tilde{w}_x f\left(\frac{\pi}{n+1}\right) \sum_{r=0}^n a_{n,r} \leq \sum_{r=0}^n a_{n,r} \tilde{w}_x f\left(\frac{\pi}{r+1}\right) \\
&\ll \sum_{r=0}^n a_{n,r} \left[\frac{1}{r+1} \sum_{k=0}^r \tilde{w}_x f\left(\frac{\pi}{k+1}\right) \right]
\end{aligned}$$

and as above

$$|J_2| \ll \sum_{r=0}^n a_{n,r} \left[\frac{1}{r+1} \sum_{k=0}^r \tilde{w}_x f\left(\frac{\pi}{k+1}\right) \right].$$

■

4.2 Proof of Theorems 2

Theorem 2 is special case of Theorem 1, therefore, analogously as in [2, Proof of Theorem 2], we can immediately obtain our estimates. ■

4.3 Proofs of Theorems 3 and 4

The proofs are similar to the above. The results follow by (2) and (9). ■

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